Let us denote r_t the return of an asset or a portfolio of assets at time t. The *ex-ante* VaR for a $\alpha\%$ coverage rate noted $VaR_{t|t-1}(\alpha)$, anticipated conditionally to an information set, Ω_{t-1}, available at time $t-1$ is defined by the following relation:

$$\Pr[r_t < VaR_{t|t-1}(\alpha)] = \alpha \tag{1}$$

Let $I_t(\alpha)$ be the indicator variable associated to the *ex-post* observation of a $\alpha\%$ VaR violation at time t.

$$I_t(\alpha) = \begin{cases} 1 & \text{if } r_t < VaR_{t|t-1}(\alpha) \\ 0 & \text{else} \end{cases} \tag{2}$$

Christoffersen (1998) shows that the problem of VaR validity corresponds to the problem of knowing whether the violation sequence $\{I_t\}_{t=1}^{T}$ confirms the following two hypotheses or not:

- **The unconditional coverage hypothesis**: the probability of an *ex-post* loss exceeding VaR forecasts must exactly be equal to the α coverage rate:

$$\Pr\left[I_t(\alpha) = 1\right] = E\left[I_t(\alpha)\right] = \alpha \tag{3}$$

- **The independence hypothesis**: VaR violations observed at two different dates for the same coverage rate must be distributed independently. Formally, variable $I_t(\alpha)$ associated to the VaR violation at time t for a $\alpha\%$ coverage rate is independent from variable $I_{t-k}(\alpha)$, $\forall k \neq 0$. In other words, it means that past VaR violations do not hold information on current and future violations. This property is also valid for any variable belonging to the Ω_{t-1} information set available at time $t-1$. In particular, variable $I_t(\alpha)$ must be independent from past returns, past values of VaR, and also VaR violations associated to any other coverage rate $\beta \in \,]0,1[$, *i.e.* $I_{t-k}(\beta)$, $\forall k \neq 0$.

Let us finally indicate that these two hypotheses (unconditional coverage and independence) are satisfied when the process associated to VaR violations is a martingale difference, that is when:

$$E\left[I_t(\alpha) - \alpha \mid \Omega_{t-1}\right] = 0 \tag{4}$$

where information set Ω_{t-1} can include not only past VaR violations defined for the $\alpha\%$ reference rate, *i.e.* $\{I_{t-1}(\alpha), I_{t-2}(\alpha), ...\}$, but also any variable Z_{t-k} known at time $t-1$, such as past VaR levels, returns, but also the violations associated to any other coverage rate β, *i.e.* $\{I_{t-1}(\beta), I_{t-2}(\beta), ...\}$. Let us keep in mind that the martingale difference property indeed implies that $\forall Z_{t-k} \in \Omega_{t-1}$, $E\left[(I_t(\alpha) - \alpha) \otimes Z_{t-k}\right] = 0$ and in particular that if $I_{t-k}(\beta) \in \Omega_{t-1}$, then

$$E\left\{\left[I_t(\alpha) - \alpha\right]\left[I_{t-k}(\beta) - \beta\right]\right\} = 0, \quad \forall(\alpha,\beta) \quad \forall k \neq 0 \tag{5}$$

Here we find the independence property again, whereas the unconditional coverage property stems from the property of iterated expectations, because the null conditional moment $E\left[I_t(\alpha) - \alpha \mid \Omega_{t-1}\right] = 0$ implies the null unconditional moment and so consequently the equality $E\left[I_t(\alpha)\right] = \alpha$.

Christoffersen (1998) supposes that, under the alternative hypothesis of VaR inefficiency, the process of $I_t(\alpha)$ violations is modeled with a Markov chain whose matrix of transition probabilities is defined by:

$$\Pi = \begin{pmatrix} \pi_{00} & \pi_{01} \\ \pi_{10} & \pi_{11} \end{pmatrix} \tag{6}$$

where $\pi_{ij} = Prob\left[I_t(\alpha) = j \mid I_{t-1}(\alpha) = i\right]$. This Markov chain shows/reflects the existence of a order one memory in the process $I_t(\alpha)$: the probability of having a violation (resp. not having one) for the current period depends on the occurrence or not of a violation (for the same level of coverage α) in the previous period. The null hypothesis of conditional efficiency is then defined by the following equality:

$$H_{0,CC}: \ \Pi = \Pi_\alpha = \begin{pmatrix} 1-\alpha & \alpha \\ 1-\alpha & \alpha \end{pmatrix} \tag{7}$$

If we do not reject the null hypothesis, then we accept the unconditional coverage hypothesis. Whatever the state of the system in $t-1$, the probability of having a violation at time t is equal to the α, coverage rate, *i.e.* $\pi_t = \Pr\left[I_t(\alpha) = 1\right] = \alpha$. Furthermore, the probability of a violation at time t is independent from the state in $t-1$. A simple likelihood ratio statistic, denoted LR_{CC}, then allows to test the null hypothesis of conditional efficiency. Under H_0, Christoffersen shows that:

$$LR_{CC} = -2\left\{\ln L\left[\Pi_\alpha, I_1(\alpha), .., I_T(\alpha)\right] - \ln L\left[\widehat{\Pi}, I_1(\alpha), .., I_T(\alpha)\right]\right\} \xrightarrow[T\to\infty]{L} \chi^2(2)$$

where $\widehat{\Pi}$ is the maximum likelihood estimator of the transition matrix under the alternative hypothesis and where $\ln L\left[\Pi, I_1(\alpha), .., I_T(\alpha)\right]$ is the log-likelihood of violations $I_t(\alpha)$ associated to a transition matrix Π, such as:

$$L\left[\Pi, I_1(\alpha), .., I_T(\alpha)\right] = (1-\pi_{01})^{n_{00}} \pi_{01}^{n_{01}} (1-\pi_{11})^{n_{10}} \pi_{11}^{n_{11}} \tag{8}$$

with n_{ij} the number of times we have $I_t(\alpha) = j$ and $I_{t-1}(\alpha) = i$.

Under the null of independence, we have:

$$H_{0,IND}: \ \Pi = \Pi_\pi = \begin{pmatrix} 1-\pi & \pi \\ 1-\pi & \pi \end{pmatrix} \tag{9}$$

where the probability π is not necessarily equal to the coverage rate α. The corresponding statistic is:

$$LR_{IND} = -2\left\{\ln L\left[\widehat{\Pi}_\pi, I_1(\alpha), .., I_T(\alpha)\right] - \ln L\left[\widehat{\Pi}, I_1(\alpha), .., I_T(\alpha)\right]\right\} \xrightarrow[T\to\infty]{L} \chi^2(1) \tag{10}$$

whith

$$\widehat{\Pi}_{\pi} = \begin{pmatrix} 1-\widehat{\pi} & \widehat{\pi} \\ 1-\widehat{\pi} & \widehat{\pi} \end{pmatrix} \tag{11}$$

and $\widehat{\pi} = \sum I_t/T$.