1 Disequilibrium econometrics: the short-side rule

Since Fair and Jaffe (1972) a large body of literature has been devoted to the econometric problems associated with estimating demand and supply schedules in disequilibrium markets. The main approach consists in using some maximum likelihood (ML) methods. In a seminal paper, Maddala and Nelson (1974) derived the general likelihood function for different disequilibrium models and proposed the appropriate ML estimating procedures. The simplest model considered by the authors is as follows:

\[ d_t = x'_{1,t} \beta_1 + \varepsilon_{1,t} \] (1)
\[ s_t = x'_{2,t} \beta_2 + \varepsilon_{2,t} \] (2)
\[ q_t = \min (d_t, s_t) \] (3)

where \( d_t \) denotes the unobservable quantity demanded during period \( t \), \( s_t \) the unobservable quantity supplied during period \( t \), \( x'_{1,t} = \left( x^{(1)}_{1,t} x^{(1)}_{2,t} \ldots x^{(1)}_{K_1,t} \right)' \) is a vector of \( K_1 \) explanatory variables that influence \( d_t \), \( x'_{2,t} = \left( x^{(2)}_{1,t} x^{(2)}_{2,t} \ldots x^{(2)}_{K_2,t} \right)' \) is a vector of \( K_2 \) explanatory variables that influence \( s_t \), \( \beta_1 \) and \( \beta_2 \) are respectively \((K_1, 1)\) and \((K_2, 1)\) vectors of parameters. We assume that \( d_t \) and \( s_t \) are unobservable at date \( t \), whereas \( x_{1,t} \) and \( x_{2,t} \) are observable. The variable \( q_t \) denotes the actual quantity observed at time \( t \). The equation (3) is the crucial disequilibrium hypothesis, which allows for the possibility that the price of the exchanged good is not perfectly flexible and rationing occurs. More generally, equation (3) indicates that any disequilibrium which takes place, i.e. any divergence between the quantity supplied and demanded, results from lack of complete price adjustment. Therefore, on the basis of voluntary exchange the “short side” of the market prevails.

Because of the equation (3), the model itself determines the probabilities with which each observation belongs to either supplied or demanded quantities. Following this,
we briefly develop the theoretical underpinnings of this result. In a first version of the model, Maddala and Nelson (1974) assume that both residuals \( \varepsilon_{1,t} \) and \( \varepsilon_{2,t} \) are stationary processes, independently and normally distributed with variance \( \sigma_1^2 \) and \( \sigma_2^2 \) respectively. Under these regularity assumptions, the transformed variable \( \varepsilon_{1,t} - \varepsilon_{2,t} \) is normally distributed with a variance equal to \( \sigma_1^2 + \sigma_2^2 \). Hence, the reduced variable \( (\varepsilon_{1,t} - \varepsilon_{2,t}) / \sigma \) follows a \( N(0,1) \). Then, the probability that the observation \( q_t \) belongs to the demand regime, denoted \( \pi_t^{(d)} \), can be computed as the corresponding \( N(0,1) \) fractile:

\[
\pi_t^{(d)} = P(D_t < S_t) = \Phi(h_t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{h_t} e^{-x^2/2} dx
\]  

(4)

where \( h_t = (x_{2,t}\beta_2 - x_{1,t}\beta_1) / \sigma \), and \( \Phi(x) \) denotes the cumulative distribution function of the \( N(0,1) \). Symmetrically, the probability of obtaining the supply regime, denoted \( \pi_t^{(s)} \), is defined by \( P(S_t < D_t) = 1 - \Phi(h_t) \).

Let \( \theta \) denote the vector of structural parameters \( \theta = (\beta_1, \beta_2, \sigma_1, \sigma_2)^T \). In order to compute the marginal density, \( f_{Q_t}(q_t) \), of the observable variable \( q_t \), we consider the joint density of \( d_t \) and \( s_t \), denoted \( g_{d_t,s_t}(d_t, s_t) \). Given the definition of the disequilibrium, we know that:

\[
f_{Q_t}(q_t) = f_{Q_t|D_t < S_t}(q_t) + f_{Q_t|S_t < D_t}(q_t)
\]

(5)

Then, we obtain the corresponding marginal density of \( q_t \) on the two subsets (cf. Appendix (2)):

\[
f_{Q_t|D_t < S_t}(q_t) = \int_{q_t = d_t}^{\infty} g_{D_t,S_t}(d_t, z) dz
\]

(6)

\[
f_{Q_t|S_t < D_t}(q_t) = \int_{q_t = s_t}^{\infty} g_{D_t,S_t}(z, s_t) dz
\]

(7)

Finally, we obtain the unconditional density function of \( Q_t \):

\[
f_{Q_t}(q_t) = f_{Q_t}(q_t, \theta) = \int_{q_t}^{\infty} g_{D_t,S_t}(q_t, z) dz + \int_{q_t}^{\infty} g_{D_t,S_t}(z, q_t) dz
\]

(8)

Next, conditionally to a structural parameters set \( \theta \) and a sample of observable variables \( q_t, x_{1,t} \) and \( x_{2,t} \) observed on \( T \) periods, the log-likelihood function of the model is then defined by:

\[
L(\theta) = \sum_{t=1}^{T} \log[f_{Q_t}(q_t, \theta)]
\]

(9)

If we assume that both residuals \( \varepsilon_1 \) and \( \varepsilon_2 \) are independent, the unconditional density function of \( Q_t \) can be expressed as follows:

\[
f_{Q_t}(q_t) = \frac{1}{\sigma_1} \phi \left( \frac{x_{1,t}\beta_1 - q_t}{\sigma_1} \right) \Phi \left( \frac{x_{2,t}\beta_2 - q_t}{\sigma_2} \right) + \frac{1}{\sigma_2} \phi \left( \frac{x_{2,t}\beta_2 - q_t}{\sigma_2} \right) \Phi \left( \frac{x_{1,t}\beta_1 - q_t}{\sigma_1} \right)
\]

(10)

where \( \phi(.) \) denotes the normal \( N(0,1) \) density function. The proof is provided in Appendix (2). In this case, the first and second order derivatives of \( L(\theta) \) can be
computed analytically (Maddala and Nelson, 1974) or numerically. We can use an iterative procedure such as the Newton-Raphson procedure to obtain the ML estimates of the structural parameters \( \theta \). Given the estimated values of the parameters, we can compute the estimated probability that the observation \( q_t \) belongs either to the demand or the supply regime, \( \hat{\pi}_t^{(d)} \) and \( \hat{\pi}_t^{(s)} \).

2 Marginal densities of \( Q_t \) in a stable disequilibrium model

Let us denote \( g_{D_t,S_t}(d_t,s_t) \) the joint density of \( D_t \) and \( S_t \). We know that the corresponding marginal densities of the unobservable variables \( D_t \) and \( S_t \) are defined by:

\[
 f_{D_t}(d_t) = \int_{-\infty}^{\infty} g_{D_t,S_t}(d_t,z) \, dz \quad f_{S_t}(s_t) = \int_{-\infty}^{\infty} g_{D_t,S_t}(z,s_t) \, dz \quad (11)
\]

We have to compute the marginal density of \( Q_t \) on the two sub-set \( Q_t = D_t \), with \( D_t < S_t \) and \( Q_t = S_t \), with \( S_t < D_t \). When \( D_t < S_t \), for a given realization \( d_t \) of \( D_t \), the marginal density of \( Q_t \), is given by the area defined by the joint density \( g_{D_t,S_t}(d_t,z) \), for values \( z \) of \( S_t \) superior to \( d_t \). Under the assumption that \( D_t < S_t \), the marginal density of \( Q_t \) is then given by:

\[
 f_{Q_t|D_t<Z_t}(q_t) = \int_{q_t=d_t}^{\infty} g_{D_t,S_t}(d_t,z) \, dz \quad (12)
\]

Symmetrically, we obtain the marginal density of \( Q_t \) when \( S_t < D_t \).

\[
 f_{Q_t|Z_t<S_t}(q_t) = \int_{q_t=s_t}^{\infty} g_{D_t,S_t}(z,s_t) \, dz \quad (13)
\]

In the general case, we know that the marginal density of \( Q_t \) is given by:

\[
 f_{Q_t}(q_t) = \int_{q_t}^{\infty} g_{D_t,S_t}(q_t,z) \, dz + \int_{q_t}^{\infty} g_{D_t,S_t}(z,q_t) \, dz \quad (14)
\]

Let us assume that both residuals \( \varepsilon_1 \) and \( \varepsilon_2 \) are independent \( (\sigma_{12} = 0) \). In this case, the joint density can be expressed as the following simple expression:

\[
 g_{D_t,S_t}(d_t,s_t) = \frac{1}{2\pi\sigma_1\sigma_2} \exp\left\{ \frac{-1}{2} \left( \frac{d_t - x_{1,t}\beta_1}{\sigma_1} \right)^2 \right\} \times \frac{1}{2\pi\sigma_1\sigma_2} \exp\left\{ \frac{-1}{2} \left( \frac{s_t - x'_{2,t}\beta_2}{\sigma_2} \right)^2 \right\} \quad (15)
\]

Now, consider the first member of the marginal density of \( Q_t \) (equation (14)):

\[
 \int_{q_t}^{\infty} g_{D_t,S_t}(q_t,z) \, dz = \frac{1}{2\pi\sigma_1\sigma_2} \int_{q_t}^{\infty} \left\{ \exp\left[ \frac{-1}{2} \left( \frac{q_t - x'_{1,t}\beta_1}{\sigma_1} \right)^2 \right] \times \exp\left[ \frac{-1}{2} \left( \frac{z - x'_{2,t}\beta_2}{\sigma_2} \right)^2 \right] \right\} \, dz
\]

\[
 = \frac{1}{\sqrt{2\pi}\sigma_1} \exp\left[ \frac{-1}{2} \left( \frac{q_t - x'_{1,t}\beta_1}{\sigma_1} \right)^2 \right] \times \frac{1}{\sqrt{2\pi}\sigma_2} \int_{q_t}^{\infty} \exp\left[ \frac{-1}{2} \left( \frac{z - x'_{2,t}\beta_2}{\sigma_2} \right)^2 \right] \, dz
\]
In the first term of this expression, we recognize the value of the $N(0, 1)$ density function at the particular point $(q_t - x_{1,t}^2\beta_1) / \sigma_1$. Indeed:

$$\frac{1}{\sqrt{2\pi}} \exp \left[ -\frac{1}{2} \left( \frac{q_t - x_{1,t}^2\beta_1}{\sigma_1} \right)^2 \right] = \phi \left( \frac{q_t - x_{1,t}^2\beta_1}{\sigma_1} \right)$$  \hspace{1cm} (16)

where $\phi(.)$ denotes the $N(0, 1)$ density function. Since this function is symmetric the first member of the marginal density of $Q_t$ can be expressed as:

$$\int_{q_t}^{\infty} g_{D_t, S_t}(q_t, z) \, dz = \frac{1}{\sigma_1} \phi \left( \frac{x_{1,t}^2\beta_1 - q_t}{\sigma_1} \right) \times \frac{1}{\sqrt{2\pi}\sigma_2} \int_{q_t}^{\infty} \exp \left[ -\frac{1}{2} \left( \frac{z - x_{2,t}^2\beta_2}{\sigma_2} \right)^2 \right] \, dz$$  \hspace{1cm} (17)

The second term of this expression can be transformed in order to introduce the $N(0, 1)$ cumulative distribution function, denoted $\Phi(.)$. Indeed, let us consider the following change in variable $\tilde{z} = (z - x_{2,t}^2\beta_2) / \sigma_2$, with $dz = d\tilde{z} / \sigma_2$. Then, we have:

$$\frac{1}{\sqrt{2\pi}\sigma_2} \int_{q_t}^{\infty} \exp \left[ -\frac{1}{2} \left( \frac{z - x_{2,t}^2\beta_2}{\sigma_2} \right)^2 \right] \, dz = \frac{1}{\sqrt{2\pi}\sigma_2} \int_{\tilde{q}_t}^{\infty} \exp \left( -\frac{\tilde{z}^2}{2} \right) \, d\tilde{z} \sigma_2$$  \hspace{1cm} (18)

with $\tilde{q}_t = (q_t - x_{2,t}^2\beta_2) / \sigma_2$. Then, this integral can be expressed as a function $\Phi(.)$.

$$\frac{1}{\sqrt{2\pi}\sigma_2} \int_{q_t}^{\infty} \exp \left[ -\frac{1}{2} \left( \frac{z - x_{2,t}^2\beta_2}{\sigma_2} \right)^2 \right] \, dz = 1 - \Phi(\tilde{q}_t) = \Phi(-\tilde{q}_t)$$  \hspace{1cm} (19)

Finally, we obtain:

$$\int_{q_t}^{\infty} g_{D_t, S_t}(q_t, z) \, dz = \frac{1}{\sigma_1} \phi \left( \frac{x_{1,t}^2\beta_1 - q_t}{\sigma_1} \right) \Phi \left( \frac{x_{2,t}^2\beta_2 - q_t}{\sigma_2} \right)$$  \hspace{1cm} (20)

Symmetrically, we can compute the second term of the marginal density of $Q_t$ (equation (14)) as:

$$\int_{q_t}^{\infty} g_{D_t, S_t}(z, q_t) \, dz = \frac{1}{\sigma_2} \phi \left( \frac{x_{2,t}^2\beta_2 - q_t}{\sigma_2} \right) \Phi \left( \frac{x_{1,t}^2\beta_1 - q_t}{\sigma_1} \right)$$  \hspace{1cm} (21)

Then, the marginal density of $Q_t$ is defined by equation (10):

$$f_{Q_t}(q_t) = \frac{1}{\sigma_1} \phi \left( \frac{x_{1,t}^2\beta_1 - q_t}{\sigma_1} \right) \Phi \left( \frac{x_{2,t}^2\beta_2 - q_t}{\sigma_2} \right) + \frac{1}{\sigma_2} \phi \left( \frac{x_{2,t}^2\beta_2 - q_t}{\sigma_2} \right) \Phi \left( \frac{x_{1,t}^2\beta_1 - q_t}{\sigma_1} \right)$$
3 The choice of initial conditions in the ML optimization procedure

There are various methods to obtain the initial conditions on structural parameters $\theta$ in the ML iteration. Here, we use a two-step OLS procedure. First, we consider the linear regressions of the observation $q_t$ on the exogenous variables sets in both functions: $q_t = x_{i,t}^l \gamma_i + \mu_{i,t}$, with $i = 1, 2$. Given the realizations of $\gamma_1$ and $\gamma_2$, we compute a first approximation of demand and supply, as $d_t = x_{1,t}^l \gamma_1$ and $s_t = x_{2,t}^l \gamma_2$. Even if we know that $\gamma_1$ and $\gamma_2$ are not convergent estimators of $\beta_1$ and $\beta_2$, we build two subgroups of observations. In the first subgroup, denoted by index $d$, we consider only the observations on $Q_t$, $X_{1,t}$ and $X_{2,t}$ for which we have $\widetilde{d}_t \leq \widetilde{s}_t$. In the second subgroup, we consider the observations for which we have $\widetilde{s}_t < \widetilde{d}_t$. The second step of the procedure consists in applying the OLS on both subgroups:

$$q_t^{(d)} = x_{1,t}^{(d)} \beta_1 + \mu_{1,t} \quad \text{and} \quad q_t^{(s)} = x_{2,t}^{(s)} \beta_2 + \mu_{2,t}$$ (22)

Then, we use the OLS estimates $\beta_1$ as starting values for $\beta_i$ in the ML iteration. For the parameters $\sigma_1$ and $\sigma_2$, we adopt the following starting values:

$$\widetilde{\sigma}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} \hat{\mu}_{i,j} \quad i = 1, 2$$ (23)

where $n_1$ denotes the size of the “demand” subgroup of observations for which we have $\widetilde{d}_t \leq \widetilde{s}_t$, and $n_2$ denotes the size of the corresponding “supply” subgroup. Some Monte Carlo simulations of the accuracy of this procedure are available upon request.

References


